Differentiable Welfare Theorems
Existence of a Competitive Equilibrium: Preliminaries

Econ 3030 Fall 2025

Lecture 22

Outline

1. Welfare Theorems in the differentiable case
2. Aggregate excess demand and existence

Welfare Theorems in the Differentiable Case

Question

What is the relationship between the first order conditions that correspond to a competitive equilibrium and those that give Pareto optimality?

Make the following assumptions

- Consumers
  - Let $X_i = \mathbb{R}^L_+$ and assume there exist $u_i(\mathbf{x})$ representing $\succsim_i$ for each i.
  - Normalize things so that $u_i(\mathbf{0}) = 0$.
  - Assume each $u_i(\mathbf{x})$ is twice continuously differentiable, with $\nabla u_i(\mathbf{x}) \gg 0$ for any $\mathbf{x}$, and also assume that $u_i(\mathbf{x})$ is quasi-concave;
    - this means preferences satisfy strong monotonicity and convexity.
- Producers
  - Production sets are $Y_j = \{\mathbf{y} \in \mathbb{R}^L : F_j(\mathbf{y}) \leq 0\}$, where $F_j(\mathbf{y}) = 0$ defines the transformation frontier.
  - Assume each $F_j(\mathbf{y})$ is convex, twice continuously differentiable, with $\nabla F_j(\mathbf{y}) \gg 0$ for any $\mathbf{y}$, and also assume that $F_j(\mathbf{0}) \leq 0$.

Welfare Theorems in the Differentiable Case

- Under these assumptions, Pareto efficiency solves the planner's problem.

Remark

An allocation is Pareto optimal if and only if it is a solution to the following:

$$\max_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{LI}\times\mathbb{R}^{LJ}} u_1\left(x_{11}, x_{21}, ..., x_{L1}\right)$$

subject to

$$u_i\left(x_{1i}, x_{2i}, ..., x_{Li}\right) \geq \bar{v}_i \qquad i = 2, 3, ..., I$$

$$\sum_i x_{li} \leq \sum_{i=1}^{I} \omega_{li} + \sum_j y_{lj} \qquad l = 1, 2, ..., L$$

$$F_j\left(y_{1j}, ..., y_{Lj}\right) \leq 0 \qquad j = 1, 2, ..., J$$

Welfare and Equilibirum In the Differentiable Case

$(\hat{x}, \hat{y})$ is Pareto optimal if and only if it solves the following

$$\max_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{L(I+J)}} u_1(\mathbf{x}_1) \quad s.t. \quad \begin{array}{ll} u_i(\mathbf{x}_i) \geq \bar{v}_i & i = 2, ..., I \\ F_j(\mathbf{y}_j) \leq 0 & j = 1, 2, ..., J \\ \sum_i x_{li} \leq \sum_{i=1}^{I} \omega_{li} + \sum_j y_{lj} & l = 1, 2, ..., L \end{array}$$

- We can also write the maximization problems that must be solved by a competitive equilibrium.

A competitive equilibrium with transfers x^*, y^*, p^* solves the following $I + J$ optimization problems (with $w_i = p^* x_i^*$ for all i):

$$\max_{\mathbf{x}_i \geq 0} u_i(\mathbf{x}_i) \quad s.t. \quad \mathbf{p}^* \cdot \mathbf{x}_i \leq w_i \quad i = 1, 2, ..., I$$

and

$$\max_{\mathbf{y}_j} \mathbf{p}^* \cdot \mathbf{y}_j \quad s.t. \quad F_j(\mathbf{y}_j) \leq 0 \quad j = 1, 2, ..., J$$

- What is the connection between the first order conditions of these two optimization problems?

Welfare and Equilibrium In the Differentiable Case

$(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal if and only if it solves the following

$$\max_{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{L(I+J)}} u_1(\mathbf{x}_1) \quad s.t. \quad \begin{array}{cc} u_i(\mathbf{x}_i) \geq \bar{u}_i & i = 2, ..., I \\ F_j(\mathbf{y}_j) \leq 0 & j = 1, 2, ..., J \\ \sum_{i=1}^{I} x_{li} \leq \sum_{i=1}^{I} \omega_{li} + \sum_{j=1}^{J} y_{lj} & l = 1, 2, ..., L \end{array}$$

The *first order (necessary and sufficient) conditions for a Pareto optimum* are:

$$x_{li} \quad : \quad \delta_i \frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_{li}} - \mu_l \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \qquad \textit{for all } i \textit{ and } l$$

$$y_{lj} \quad : \quad \mu_l - \gamma_j \frac{\partial F_j(\mathbf{y}_j^*)}{\partial y_{lj}} = 0 \qquad \textit{for all } j \textit{ and } l$$

- δ_is are I Lagrange multiplier associated with i's utility constraint
  - $\delta_1 = 1$ to make the notation more compact.
- μ_ls are L Lagrange multiplier associated with good l feasibility constraint.
- γ_j are J Lagrange multiplier associated with j's technology constraint.

Pareto Optimality In The Differentiable Case

The first order conditions for a Pareto optimum are:

$$x_{li} \quad : \quad \delta_i \frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_{li}} - \mu_l \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \qquad \text{for all } i \text{ and } l$$

$$y_{lj} \quad : \quad \mu_l - \gamma_j \frac{\partial F_j(\mathbf{y}_j^*)}{\partial y_{lj}} = 0 \qquad \text{for all } j \text{ and } l$$

where, to make the notation more compact, $\delta_1 = 1$.

Interpret the multipliers as follows:

- μ_l is the change in *consumer 1* utility from a marginal increase in the aggregate endowment of good l (this the "shadow price" of good l).
- δ_i is the change in *consumer 1* utility from a marginal decrease in the utility of consumer i.
- γ_j is the marginal cost of tightening production constraint j.
- Thus $\gamma_j \frac{\partial F_j}{\partial y_{lj}}$ is the marginal cost of increasing y_{lj}; it must be equal to the marginal benefit of good l (measured in consumer 1 utility).

Equilibrium in the Differentiable Case

A price equilibrium with transfers solves

$$\max_{\mathbf{x}_i \geq 0} u_i(\mathbf{x}_i) \text{ s.t. } \mathbf{p}^* \cdot \mathbf{x}_i \leq w_i \; i = 1, \ldots, I \quad \text{and} \quad \max_{\mathbf{y}_j} \mathbf{p}^* \cdot \mathbf{y}_j \text{ s.t. } F_j(\mathbf{y}_j) \leq 0 \; j = 1, \ldots, J$$

The first order (necessary and sufficient) conditions for a Competitive Equilibrium are:

$$x_{li} : \frac{\partial u_i}{\partial x_{li}} - \alpha_i p_l^* \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \forall i, l \qquad y_{lj} : p_l^* - \beta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- α_i are the I Lagrange multiplier associated with i's budget constraint.
- β_j are the J Lagrange multiplier associated with j's technology constraint.

Interpret the multipliers as follows:

- α_i is the change in consumer i utility from a marginal increase in her income.
- β_j is the change in firm j profits from a marginal change in the "efficiency" of good l.

Welfare Theorems in the Differentiable Case

Pareto optimality:

- The first order conditions evaluated at a Pareto efficient allocation are (with $\delta_1 = 1$):

$$x_{li} : \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \forall i, l \qquad \text{and} \qquad y_{lj} : \mu_l - \gamma_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

Competitive equilibrium (with transfers):

- The first order necessary (and sufficient) conditions evaluated at $(\mathbf{x}^*, \mathbf{y}^*)$ yield:

$$x_{li} : \frac{\partial u_i}{\partial x_{li}} - \alpha_i p_l^* \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \forall i, l \qquad y_{lj} : p_l^* - \beta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- If one has

$$\boxed{\mu_l = p_l^*, \qquad \delta_i = \frac{1}{\alpha_i}, \text{ and } \qquad \gamma_j = \beta_j}$$

then solutions to the Pareto efficiency equations also solve the competitive equilibrium equations and vice-versa.

- In other words the two welfare theorems hold.

Welfare Theorems and "Marginal Rates" Conditions

- A way to look at the implications of the first order conditions of the different maximization problems is to think about consumers' marginal rate of substitutions and firms' marginal rates of transformations.
- These are easier to see if one focuses on interior Pareto optimal allocations and interior competitive equilibria.
- From now on, focus on the case in which the solution to the maximization problems have the feature that $x_{li} > 0$ and $y_{lj} > 0$ for all l, all i, and all j.

"Marginal" Conditions for Pareto Optima

The first order conditions (with $\delta_1 = 1$) for a Pareto optimum at an interior solution are:

$$x_{li} : \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l = 0 \quad \forall i, l \qquad \textit{and} \qquad y_{lj} : \mu_l - \gamma_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- For any two goods l and l', the consumers' FOC implies

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}}$$

  where the left hand side does not depend on the consumer.

MRS between any two goods must be equal across any two consumers:

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}} = \frac{\frac{\partial u_{i'}}{\partial x_{li'}}}{\frac{\partial u_{i'}}{\partial x_{l'i'}}} \qquad \text{for all } i, i' \text{ and } l, l'$$

"Marginal" Conditions for Pareto Optima

The first order conditions (with $\delta_1 = 1$) for a Pareto optimum at an interior solution are:

$$x_{li} : \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l = 0 \quad \forall i, l \qquad \textit{and} \qquad y_{lj} : \mu_l - \gamma_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- For any two goods l and l', the firms' FOC implies

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial F_{j'}}{\partial y_{lj'}}}{\frac{\partial F_{j'}}{\partial y_{l'j'}}}$$

  where the left hand side does not depend on the firm.

MRT between any two goods must be equal across any two firms:

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial F_j}{\partial y_{lj}}}{\frac{\partial F_j}{\partial y_{l'j}}} = \frac{\frac{\partial F_{j'}}{\partial y_{lj'}}}{\frac{\partial F_{j'}}{\partial y_{l'j'}}} \qquad \text{for all } j, j' \text{ and } l, l'$$

“Marginal” Conditions for Pareto Optima

The first order conditions (with $\delta_1 = 1$) for a Pareto optimum at an interior solution are:

$$x_{li} : \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l = 0 \quad \forall i, l \qquad \textit{and} \qquad y_{lj} : \mu_l - \gamma_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

MRS between any two goods must be equal across any two consumers:

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}} = \frac{\frac{\partial u_{i'}}{\partial x_{li'}}}{\frac{\partial u_{i'}}{\partial x_{l'i'}}} \qquad \text{for all } i, i' \text{ and } l, l'$$

MRT between any two goods must be equal across any two firms:

$$\frac{\mu_l}{\mu_{l'}} = \frac{\frac{\partial F_j}{\partial y_{lj}}}{\frac{\partial F_j}{\partial y_{l'j}}} = \frac{\frac{\partial F_{j'}}{\partial y_{lj'}}}{\frac{\partial F_{j'}}{\partial y_{l'j'}}} \qquad \text{for all } j, j' \text{ and } l, l'$$

MRS must be equal to MRT:

$$\frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}} = \frac{\frac{\partial F_j}{\partial y_{lj}}}{\frac{\partial F_j}{\partial y_{l'j}}} \qquad \text{for all } i, j \text{ and } l, l'$$

"Marginal" Conditions for Competitive Equilibrium

At an interior solution, the first order conditions for a competitive equilibrium are:

$$x_{li} : \frac{\partial u_i}{\partial x_{li}} - \alpha_i p_l^* = 0 \quad \forall i, l \qquad \textit{and} \qquad y_{lj} : p_l^* - \beta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

MRS between any two goods must equal the corresponding price ratio:

$$\frac{p_l^*}{p_{l'}^*} = \frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}} \qquad \text{for all } l, l' \text{ and for each } i$$

MRT between any two goods must equal the corresponding price ratio:

$$\frac{p_l^*}{p_{l'}^*} = \frac{\frac{\partial F_j}{\partial y_{lj}}}{\frac{\partial F_j}{\partial y_{l'j}}} \qquad \text{for all } l, l' \text{ and for each } j$$

MRS must be equal to MRT:

$$\frac{\frac{\partial u_i}{\partial x_{li}}}{\frac{\partial u_i}{\partial x_{l'i}}} = \frac{\frac{\partial F_j}{\partial y_{lj}}}{\frac{\partial F_j}{\partial y_{l'j}}} \qquad \text{for all } i, j \text{ and } l, l'$$

- Again, we have equalities between marginal rates of substitution and transformation.

The Other Planner's Problem

An equivalent maximization problem (also called planner's problem)

$$\max_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{L(I+J)}} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) \quad s.t. \quad \begin{array}{cc} F_j(\mathbf{y}_j) \leq 0 & j = 1, 2, ..., J \\ \sum_{i=1}^{I} x_{li} \leq \sum_{i=1}^{I} \omega_{li} + \sum_{j=1}^{J} y_{lj} & l = 1, 2, ..., L \end{array}$$

The first order (necessary and sufficient) conditions for a Pareto optimum are:

$$x_{li} : \lambda_i \frac{\partial u_i}{\partial x_{li}} - \psi_l \begin{cases} \leq 0 & \\ = 0 & \text{if } x_{li} > 0 \end{cases} \forall i, l \qquad \textit{and} \qquad y_{lj} : \psi_l - \eta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- Connect to the first planner's problem:

$$\mu_l = \frac{\psi_l}{\lambda_1}, \quad \delta_i = \frac{\lambda_i}{\lambda_1}, \text{ and } \quad \gamma_j = \frac{\eta_j}{\lambda_1}$$

- Conect to equilibrium:

$$p_l^* = \psi_l, \quad \alpha_i = \frac{1}{\lambda_i}, \text{ and } \quad \beta_j = \eta_j$$

The Other Planner's Problem

The first order conditions at an interior solution:

$$x_{li} : \lambda_i \frac{\partial u_i}{\partial x_{li}} - \psi_l = 0 \quad \forall i, l \qquad \textit{and} \qquad y_{lj} : \psi_l - \eta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \quad \forall j, l$$

- Sum over all goods l: $\lambda_i \sum_{l=1}^{L} \frac{\partial u_i}{\partial x_{li}} = \sum_{l=1}^{L} \psi_l$ and $\eta_j \sum_{l=1}^{L} \frac{\partial F_j}{\partial y_{lj}} = \sum_{l=1}^{L} \psi_l$ and divide the original FOC by these $\frac{\frac{\partial u_i}{\partial x_{li}}}{\sum_{l=1}^{L} \frac{\partial u_i}{\partial x_{li}}} = \frac{\psi_l}{\sum_{l=1}^{L} \psi_l}$ and $\frac{\frac{\partial F_j}{\partial y_{lj}}}{\sum_{l=1}^{L} \frac{\partial F_j}{\partial y_{lj}}} = \frac{\psi_l}{\sum_{l=1}^{L} \psi_l}$

A new interpretation of equilibrium prices

Because r.h.s is the same for all consumers (and firms): for any i and i'

$$\frac{\frac{\partial u_i}{\partial x_{li}}}{\sum_{l=1}^{L} \frac{\partial u_i}{\partial x_{li}}} = \frac{\frac{\partial u_{i'}}{\partial x_{li'}}}{\sum_{l=1}^{L} \frac{\partial u_{i'}}{\partial x_{li'}}} = \frac{p_l^*}{\sum_{l=1}^{L} p_l^*} \quad \text{for all } l$$

where the last equality is a consequence of the Welfare Theorems

- Prices reflect the common normalized marginal utilities of consumption.

- Separating step in proof of SWT: prices equal the slopes of better-than sets.

Welfare Theorems in the Differentiable Case

Summary

In the case of a differentiable economy, we can use the first order conditions to prove the following.

- Every Pareto optimal allocation is a price equilibrium with transfers for some appropriately chosen price vector $\mathbf{p}$ and welfare transfers $\mathbf{w}$ (Second Welfare Theorem);
- Any allocation corresponding to a price equilibrium with transfers must be Pareto optimal (First Welfare Theorem).

- As an exercise, write these conclusions like formal ‘theorems’ rather than ‘observations’.

NOTE

The equilibrium price of each good is equal to that good marginal ‘value’ in the planner’s problem. Intuitively, this is a measure of the scarcity of the good.

Existence of A Competitive Equilibrium

The Existence Problem

- We have theorems about the welfare properties of a competitive equilibrium, but no result stating that such a thing exists.
- Existence is a crucial question for any theory (any statement is true of the empty set).
- The problem of proving existence of a competitive equilibrium was solved in the early 1950s.
- Is it hard to prove existence?
- What assumptions do we need?
- As it turns out, economists' search for a proof was stuck for a long time.
  - Until a mathematician came along that inspired them to look at the problem in a different way.

- To answer these questions, we start by looking at the conditions that characterize an equilibrium and see what inspiration they can give us.

Equations and Unknowns

- For some price vector p, suppose we find Walrasian demand $x_i^*(\mathbf{p})$, for each consumer, and supply $y_j^*(\mathbf{p})$, for each firm.
- We typically find an equilibrium by finding a $\mathbf{p}$ that solves the following system of L equations:

$$\begin{aligned}
\sum_{i=1}^{I}(x_{1i}^*(\mathbf{p}) - \omega_{1i}) - \sum_{j=1}^{J} y_{1j}^*(\mathbf{p}) &= 0 \\
\dots\dots\dots\dots\dots\dots& \\
\sum_{i=1}^{I}(x_{Li}^*(\mathbf{p}) - \omega_{Li}) - \sum_{j=1}^{J} y_{Lj}^*(\mathbf{p}) &= 0
\end{aligned}$$

  - In general we may have correspondences, but ignore that for now.
- We have L equations in L unknowns (the prices).
  - Need to normalize prices (they sum up to one, or price of one of the goods is equal to one).
  - Is one of the equations redundant? Yes.
- Next we focus on this system of equations.

Individual Excess Demand

Definition

The individual excess demand correspondence $z_i : \mathbb{R}^L_+ \rightarrow \mathbb{R}^L$ is:

$$z_i(\mathbf{p}) = \mathbf{x}_i^* - \boldsymbol{\omega}_i$$

where $\mathbf{x}_i^* \in x_i^*(\mathbf{p})$ for each $i = 1, ..., I$.

- For each individual, this measures the difference between that individual's demand and supply as prices change.
- This is a correspondence because Walrasian demand is multivalued.
- If Walrasian demand is single valued for any price vector (it is a function), then individual excess demand is a function.

Market Excess Demand

Definition

The market excess demand correspondence $z : \mathbb{R}^L_+ \to \mathbb{R}^L$ is:

$$z(\mathbf{p}) = \sum_{i=1}^{I} (\mathbf{x}_i^* - \omega_i) - \sum_{j=1}^{J} \mathbf{y}_j^*$$

where $\mathbf{y}_j^* \in y_j^*(\mathbf{p})$ for each $j = 1, ..., J$, and $\mathbf{x}_i^* \in x_i^*(\mathbf{p})$ for each $i = 1, ..., I$.

- For each good, this measures the difference between total (aggregate) demand and total (aggregate) supply as prices change.
- This is a correspondence because Walrasian demand is multivalued and/or supply is multivalued.
- If Walrasian demand and firms' supply are single valued for each consumer and each firm, aggregate excess demand is a function.

Properties of Market Excess Demand in Exchange Economies

Proposition

Consider an exchange economy where $\sum_{i=1}^{I} \omega_i >> 0$, and assume that for each i we have that $X^i = \mathbb{R}_+^L$, and $\succsim_i$ is continuous, strictly convex, and locally non-satiated. Then the aggregate excess demand function $z(\mathbf{p})$ satisfies the following properties:

1. *$z(\mathbf{p})$ is continuous;*
2. *$z(\mathbf{p})$ is homogeneous of degree zero;*
3. *there exists an $s > 0$ such that $z_l(\mathbf{p}) > -s$ for each l and for each $\mathbf{p}$.*

- Proof is in next Problem Set.
- Think about how these conditions are inherited from individual excess demands.

The Value of Market Excess Demand

- Let $\mathbf{p}^*$ be an equilibrium price vector, and calculate the value of market excess demand at prices $\mathbf{p}^*$.

- We have

$$\mathbf{p}^* \cdot z(\mathbf{p}^*) = \sum_{l=1}^{L} p_l^* \left[\sum_{i=1}^{I} (x_{li}^* - \omega_{li}) - \sum_{j=1}^{J} y_{lj}^* \right] = 0$$

- The second equality follows because at an equilibrium
  - either the terms in square parenthesis are 0,
  - or the corresponding price is equal to 0.

- Surprisingly, when preferences are locally non satiated, the value of market excess demand is zero at any price vector.

Theorem (Walras' Law)

In an economy where $\succsim_i$ *is locally non-satiated for all* $i = 1, ..., I$*:*

$$\mathbf{p} \cdot z(\mathbf{p}) = 0 \qquad \textit{for all } \mathbf{p}$$

Proof.

For a fixed price vector $\mathbf{p}$, take any $\mathbf{y}_j^* \in y_j^*(\mathbf{p})$ and $\mathbf{x}_i^* \in x_i^*(\mathbf{p})$

$$\begin{aligned}
\mathbf{p} \cdot z(\mathbf{p}) &= \mathbf{p} \cdot \sum_{i=1}^{I} (\mathbf{x}_i^* - \omega_i) - \mathbf{p} \cdot \sum_{j=1}^{J} \mathbf{y}_j^* = \sum_{i=1}^{I} (\mathbf{p} \cdot \mathbf{x}_i^* - \mathbf{p} \cdot \omega_i) - \sum_{j=1}^{J} \mathbf{p} \cdot \mathbf{y}_j^* \\
&\underbrace{=}_{by\ l.n.s.} \sum_{i=1}^{I} \left(\mathbf{p} \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \mathbf{p} \cdot \mathbf{y}_j^* - \mathbf{p} \cdot \omega_i \right) - \sum_{j=1}^{J} \mathbf{p} \cdot \mathbf{y}_j^* \\
&= \sum_{j=1}^{J} \sum_{i=1}^{I} \theta_{ij} \mathbf{p} \cdot \mathbf{y}_j^* - \sum_{j=1}^{J} \mathbf{p} \cdot \mathbf{y}_j^* \underbrace{=}_{\sum_{i=1}^{I} \theta_{ij} = 1} 0
\end{aligned}$$

□

Consequence of Walras' Law

Theorem (Walras' Law)

In an economy with locally non satisted preference

$$\mathbf{p} \cdot z(\mathbf{p}) = 0 \qquad \textit{for all } \mathbf{p}$$

- The value of aggregate excess demand is zero for any price vector.

If $L-1$ markets clear, the L-th market also clears

- If the equilibrium conditions are satisfied in all but one market, they are also satisfied in the last market.
- The equilibrium conditions can be written as $p_l^* z_l(\mathbf{p}^*) = p_l^* \left[\sum_{i=1}^I (x_{li}^* - \omega_{li}) - \sum_{j=1}^J y_{lj}^*\right] = 0$ for each l.
  - Why? Because either $\sum_{i=1}^I (x_{li}^* - \omega_{li}) - \sum_{j=1}^J y_{lj}^* = 0$ or $p_l^* = 0$.
- Suppose $p_l z_l(\mathbf{p}) = 0$ for all goods but k. By Walras' Law:

$$0 = \sum_{l=1}^{L} p_l z_l(\mathbf{p}) = \sum_{l \neq k} p_l z_l(\mathbf{p}) + p_k z_k(\mathbf{p}) = 0 + p_k z_k(p)$$

Excess Demand and Competitive Equilibirum

- A competitive equilibrium says: everyone maximizes (conditions 1. and 2.) and
$$\sum_{i=1}^{I}\left(\mathbf{x}_i^* - \boldsymbol{\omega}_i\right) - \sum_{j=1}^{J}\mathbf{y}_j^* \leq 0 \qquad \text{and } p_l^* = 0 \text{ if the inequality for good } l \text{ is strict}$$

- One can re-state these conditions using market excess demand:
$$z\left(\mathbf{p}^*\right) \leq 0 \qquad \text{and } p_l^* = 0 \text{ if the inequality for good } l \text{ is strict}$$

- Hence, we have a competitive equilibrium if we find a price vector such that market excess demand is weakly negative, and prices are zero whenever market excess demand is strictly negative.

Monotonicity and Zero Prices

- If preferences are strongly monotone (this is more restrictive than local non satiation), then an equilibrium price vector must be strictly positive: if not, consumers would demand an infinite amount of any free good.

With strong monotonicity, a price vector is an equilibrium if and only if it 'clears all markets'

- Formally, a price vector $\mathbf{p}^*$ is an equilibrium if and only

$$z_l(\mathbf{p}^*) = 0 \qquad \text{for all } l,$$

  - this is a system of L equations in L unknowns.

Equilibrium in an Edgeworth Box Economy

- Let $p_2 = 1$, so market excess demand is $z(p_1, 1) = (z_1(p_1, 1), z_2(p_1, 1))$
  - assume these are functions (strictly convex preferences).
- If $z_1(p_1^*, 1) = 0$ by Walras' Law the other market also clears: an equilibrium exists if there is a price of good one such that $z_1(p_1^*, 1) = 0$.

In an Edgeworth Box Economy

Showing an equilibrium exists is like showing that some function has a zero.

- Intermediate Value Theorem: a continuous function that is negative and positive must have a zero.
- With more than 2 goods we need a more general version of this idea.

Next Class

- Existence (of a competitive equilibrium)